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Non-reflective boundary conditions and the viscous instability in accretion discs

Patrick Godon

* *California Institute of Technology*
NASA/Jet Propulsion Laboratory, MS 238-S32, 4800 Oak Groun Dr., Pasadena, CA 91109

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ABSTRACT

A numerical investigation is carried out to analyze the effects of the outer boundary conditions on the viscous instability of accretion discs. A one-dimensional polytropic time dependent calculations of geometrically thin accretion discs is used. When the outer boundary is reflective, oscillations are trapped between the outer boundary and the outer edge of the inner evanescent region. When the outer boundary is non-reflective the oscillations decay on a viscous time scale. An analytical treatment of numerically non-reflective boundary conditions is carried out for a polytropic flow and for a gas including radiation.

Key words: accretion, accretion discs waves - methods: numerical - binaries: close - stars: oscillations.

1 INTRODUCTION

Kato (1978) and Blumenthal, Yang & Lin (1984) (see also Carroll *et al.* 1985 and the references therein) have carried out local analyses of the stability of accretion discs. They found that the alpha disc model admits radial pulsational unstable regions. These unstable regions are due to local variations of the viscosity. The stability criterion is strongly associated with the dependence of the viscosity on the density and temperature during oscillations: oscillations are excited when the viscosity increases in the compressed phase in comparison to the expanded phase.

A first global analysis of oscillations in accretion discs was carried out by Papaloizou and Stanley (1986), who use a polytropic approximation to study analytically and numerically the mechanism by which overstable oscillations can evolve and dissipate in accretion discs. Okuda & Mineshige (1991) and Okuda *et al.* (1992) also examine the time evolution of the radial pulsational instabilities found by Kato (1978) and Blumenthal *et al.* (1984) using a one-dimensional hydrodynamic code including radiation. The high amplitude oscillations appear in the numerical calculations only in the outer region of the disc ($r > 3R_*$). In the inner part of the disc and for low value of the alpha viscosity parameter ($\alpha < 1$), the oscillations decrease due to the propagation properties of short waves. However, when $\alpha \approx 1$, the oscillations do not decay in the inner part of the disc. Papaloizou & Stanley (1986) furthermore stress that the local instabilities are not expected to appear in the outer part of the disc, unless one considers the disc boundaries. Oscillations are ex-

pected in the outer region of the disc only if the outer boundary is reflective (for example a free or a rigid boundary). So far, no calculations were carried out with non-reflective boundary conditions, the previous authors have assumed a numerically reflective outer boundary without justifying or specifying its physical origin. The outer boundary is numerically reflective since the inflow boundary conditions are imposed directly on the primitive variables. In a realistic situation, the boundary conditions are the specification of the conditions which exist outside the computational domain. The inflowing characteristics of the flow, which propagate inward into the computational domain, carry with them the conditions through the boundary. Therefore, the boundary conditions have to be imposed on the inflowing characteristics of the flow, and not on the primitive variable. The imposition of the conditions on the inflowing characteristics of the flow leads to non-reflective boundary conditions.

In this work, we prove numerically that the local instabilities do not appear in the outer part of the disc, when a proper treatment of non-reflective boundary conditions is carried out. When one carries out a numerical investigation of the viscous instability, one has to take into account that the resulting oscillations obtained can be an artificial result of the wrong treatment of the boundary conditions. A physical reflective outer boundary, however, could be the outer edge of the disc where the density drops (free boundary), or, for example, a high density spiral arm, tidally excited by a companion star (rigid boundary).

In section 2 we present the equations and assumptions made in the physical model, together with the numerical method. In section 3 we give an analytical treatment of non-

* email: godon@herca.jpl.nasa.gov

reflective boundary conditions for a polytropic flow. The results are presented and discussed in section 4.

2 EQUATIONS AND ASSUMPTIONS

2.1 Equations

A polytropic Equation of State is assumed. The equations are written in cylindrical coordinates (r, ϕ, z) , they include the gravity of the star and a viscous term. The treatment is one-dimensional (in r). The disc and the boundary layer are assumed to be in hydrostatic equilibrium and geometrically thin in the vertical direction (2). The equations can be written for the momenta $U = \rho v_r$ and $W = \rho v_\phi = \rho r \Omega$ in the following manner:

the conservation of mass

$$\frac{\partial \rho}{\partial t} + \text{div}[\tilde{v}\rho] = 0, \quad (1)$$

the conservation of radial momentum

$$\frac{\partial U}{\partial t} + \text{div}[\tilde{v}U] - \frac{W^2}{r\rho} = -\frac{\partial P}{\partial r} - \rho \frac{\partial V}{\partial r} + F_r, \quad (2)$$

the conservation of angular momentum

$$\frac{\partial W}{\partial t} + \text{div}[\tilde{v}W] + \frac{UW}{r\rho} = F_\phi, \quad (3)$$

where

$$\text{div}[\tilde{v}f] = \frac{1}{r} \frac{\partial}{\partial r} \left[rU f \right],$$

The viscous force in the radial equation of motion is given by

$$F_r = \frac{\partial}{\partial r} \left[\frac{4}{3} \nu_r \rho \frac{\partial}{\partial r} \left(\frac{rU}{\rho} \right) \right],$$

while in the azimuthal equation of motion the viscous term is

$$F_\phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left[\nu_r r^3 \frac{\partial \Omega}{\partial r} \right].$$

V is the gravitational potential of the star:

$$V = -\frac{GM}{r},$$

In general, one defines an average density $\bar{\rho}$ through the relation $\Sigma = 2H\bar{\rho} = \int_{-H}^H \rho(z) dz$. Here the density is defined through the relation $\Sigma = \int_{-H_{max}}^H \rho(z) dz = 2\rho H_{max}$, which holds for all $H_{max} > H(z)$. Inserting $\Sigma = 2\rho H_{max}$ into the usual vertically integrated equations [see for example Savonije, Papaloizou and Lin (1994), for such equations] leads to the above set of equations. This is completely equivalent to solving the equations for the mid-plane central quantities, therefore allowing $\rho = \rho_c$. The pressure is given by:

$$P = K \rho^{1+1/n}, \quad (4)$$

where K is the polytropic constant and n is the polytropic index.

2.2 Boundary conditions

In this subsection, we present the physical boundary conditions which can be imposed on the one-dimensional flow at the boundaries. If these conditions are imposed directly on the 'primitive' variables of the flow (ρ, v_r and Ω), the boundary becomes reflective. In order to ensure a non-reflective boundary, the conditions have to be imposed on the incoming characteristics of the flow. In section 3, we carry out an analytical treatment of non-reflective boundary conditions (see Givoli 1991, for a review of non-reflective boundary conditions).

The inner boundary of the computational domain is the rotating stellar surface ($r = R_*$) through which matter flows into the star at a constant rate (\dot{M}). The outer boundary of the computational domain is the inner edge of the disc and rotates at Keplerian velocity. The gas enters this boundary at the same rate as it leaves through the inner boundary (\dot{M}).

2.3 Initial conditions

Since the equations are time dependent, initial conditions have to be specified. The initial conditions used here are the superposition of an atmosphere and an inflowing disc of matter. The initial pressure is given through the equation of state. The initial radial momentum is obtained through the relation $\dot{M} = \text{const.}$

2.4 Viscosity prescription

Since we are interested to study the viscous instability of the standard thin disc model, the standard α viscosity prescription is used for the viscosity law (Shakura & Sunyaev 1973)

$$\nu = \alpha c_s H, \quad (5)$$

where $c_s^2 = \gamma P / \rho$. In the calculations we chose $\alpha = 0.1$.

2.5 The numerical method

The spatial dependence of the equations is treated with a Chebyshev method of collocation (Gottlieb & Orszag 1977, Voigt, Gottlieb & Hussaini 1984, Canuto et al. 1988), while an explicit fourth order Runge-Kutta method is used for the time dependence of the equations. The Chebyshev spectral method was described in Godon, Regev & Shaviv (1995) and Godon (1995). The Chebyshev method is appropriate for non-periodic boundary conditions and the repartition of the gridpoints is higher at the boundaries.

In the present work, the method is improved such that the grid spacing is $\Delta x \approx 1/N^2$ at the inner boundary and $\Delta x \approx 1/N$ at the outer boundary (Gos1011 & Tal-Ezer 1993).

The Chebyshev method is further implemented by the use of Fast Fourier Transform and spectral filters.

3 THE OPEN BOUNDARY

When hyperbolic equations are solved numerically, instabilities frequently appear, *due* to the incorrect treatment of

the boundary conditions (Gottlieb, Gunzburger & Turkel 1982). Moreover, in sonic cases, the incorrect treatments of the boundary conditions, even if stable, can introduce additional periodic components to the solution (Abarbanel et al. 1991). For the treatment of the boundary conditions at an open boundary, the full Navier-Stokes equations can be considered as almost hyperbolic, since the effect of the viscosity is negligible there, while at a rigid boundary the viscous effects are dominant. In order to avoid instabilities and additional periodic phenomena, one has to impose the boundary conditions on the incoming characteristic variables rather than on the natural variables. The principle is to propagate the flow invariants along the characteristic lines through the boundary. For one-dimensional problems the flow invariants of the Euler equations are the Riemann invariants. The Riemann invariants are divided into those propagating out of the domain of computation and those carried into it. It is therefore, required to provide boundary conditions for the inflow variables, while the outflow variables are determined by their calculated values along the characteristic lines.

Therefore, the treatment of the outer boundary is carried out in the following manner. Equations (I-4) are rewritten as an homogeneous hyperbolic system:

$$\begin{cases} \frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} = 0, \\ \frac{\partial \rho}{\partial t} + \rho \frac{\partial v_r}{\partial r} + \frac{2}{r} \rho v_r = 0, \\ \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{1}{\rho} \frac{\partial P}{\partial r} = 0. \end{cases} \quad (6)$$

Making use of the relation $P = K\rho^\gamma$ ($\gamma = 1 + 1/n$), and the sound speed $c_s^2 = \gamma P/\rho$, we find:

$$\begin{cases} \frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} = 0, \\ \frac{2}{\gamma-1} \frac{\partial c_s}{\partial t} + c_s \frac{\partial v_r}{\partial r} + \frac{2}{\gamma-1} v_r \frac{\partial c_s}{\partial r} = 0, \\ \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + c_s \frac{\partial c_s}{\partial r} = 0. \end{cases} \quad (7)$$

Adding and subtracting the last two equations leads to an homogeneous hyperbolic system for the characteristics (eigenvectors) of the flow:

$$\begin{cases} \left[\frac{\partial}{\partial t} + v_r \frac{\partial}{\partial r} \right] v_\phi = 0, \quad (I) \\ \left[\frac{\partial}{\partial t} + (v_r + c_s) \frac{\partial}{\partial r} \right] (v_r - 2nc_s) = 0, \quad (II) \\ \left[\frac{\partial}{\partial t} + (v_r - c_s) \frac{\partial}{\partial r} \right] (v_r - 2nc_s) = 0, \quad (III) \end{cases} \quad (8)$$

where v_r , $v_r + c_s$ and $v_r - c_s$ are the eigenvalues associated with the eigenvectors (characteristics) v_ϕ (I), $v_r + 2nc_s$ (II), and $v_r - 2nc_s$ (III) respectively. When the flow is subsonic, the fast outflow characteristics at the outer radial boundary $r = R_0$ is (II), while the fast inflow characteristic is (III). The slow characteristics (I), is incoming when $u_r < 0$ and outgoing when $v_r > 0$. (When the flow is supersonic all the quantities (I), (II), (III) are incoming for negative radial velocity and outgoing for positive radial velocity). The boundary conditions are then imposed on the flow characteristics in the following manner: exact values from outside the boundary are imposed on the inflow variables, while values obtained from the computation at the boundary are imposed on the outflow variables. For v_r positive (and subsonic) one solves the system

$$\begin{aligned} (I) &= (I)_e, \\ (II) &= (II)_e, \\ (III) &= (III)_e, \end{aligned} \quad (9)$$

while for negative u_r (and subsonic) one solves the system

$$\begin{aligned} (I) &= (I)_e, \\ (II) &= (II)_e, \\ (III) &= (III)_e. \end{aligned} \quad (10)$$

The exact values (the standard thin Keplerian disc model) are denoted by e and the values computed at the outer boundary are denoted by c . This system of equations is then solved for the dependent variables ρ, v_r, v_ϕ . The solution for v_ϕ is trivial, while the solutions for v_r and ρ can be considered as a rectification of the boundary conditions imposed on v_r and c_s :

$$\begin{cases} v_r = (v_r)_e + \left[\frac{\Delta v_r}{2} + 2n \frac{\Delta c_s}{2} \right], \\ c_s = (c_s)_e + \left[\frac{\Delta v_r}{4n} + \frac{\Delta c_s}{2} \right], \end{cases} \quad (11)$$

with

$$\rho = \left[\frac{nc_s^2}{(n+1)K} \right]^n,$$

and $\Delta v_r = (v_r)_c - (v_r)_e$, $\Delta c_s = (c_s)_c - (c_s)_e$. The terms in the squared brackets (eq. 11) are the rectification terms. The values taken by the primitive variables at the open boundary are not especially equal to the values imposed there. In fact the above equations allow the variables to take the values which propagate through the open boundary. As the solution approaches steady state, the rectifications terms becomes negligible, and the primitive variables take the values which are imposed at the boundary (this is true when the imposed value is the steady state solution or a good approximation to it).

This particular treatment of the boundary conditions has the advantage of avoiding numerical instabilities originating from the boundary and it assures non-reflecting boundary conditions.

However, the treatment carried out here for a polytropic flow cannot be carried out for a more realistic flow. In the Appendix, we carry out an approximate treatment (a linearization of the equations) for a flow with an equation of state of the form $mP = \mathcal{R}\rho T / \mu + aT^4/3$. The equations also include the energy equation.

4 RESULTS AND DISCUSSIONS

In the work presented here, we calculate mainly two models. The models are exactly the same except for the outer boundary conditions. In model 1 we impose non-reflective boundary conditions at the outer boundary, while in model 2 we impose reflective boundary conditions.

In the two models $\nu_r = \nu$, $n = 3$ and K is chosen such that $(H/r) = 0.02$ at the outer boundary. If we define $\epsilon = n/r = c_s/r\Omega_K$, then the polytropic constant can be written

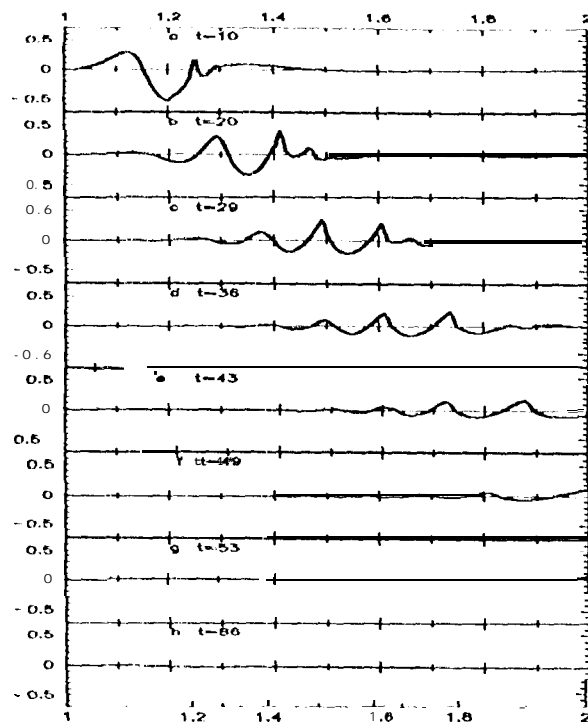


Figure 1. The Mach number v_r/c_s is shown as a function of the radius r/R_* for model 1. Eight snapshots are taken at different times to show how the model relaxes in the inner part of the disc. Here the outer non-reflective boundary has been placed at $r = 2R_*$. The initial perturbation (due to the initial condition which is only a guess of the solution) propagates outward and exits the outer boundary without any reflection or oscillations.

$$K = \frac{z_0^2 \Omega_K^2}{2(n+1)\rho_0^{1/n}} \quad (12)$$

where $Z = \sqrt{2\pi r c_{\text{aorl}} \rho_0}$ is the mid-plane value of the density (see for example Papaloizou & Stanley 1986, Bisnovatyi-Kogan 1993). The number of grid points is $N = 256$. The unit of length is R_* and the unit of time is $\Omega_K^{-1}(R_*)$. Since we are not interested to solve for the structure of the boundary layer, but rather for the oscillations in the outer part of the disc, we choose $\Omega_* = 0.98 \Omega_K$ for numerical convenience (see below). In both models $\alpha = 0.1$.

After a few dynamic times ($\Omega_K^{-1}(R_*)$) the models already approach steady state, however their evolution is followed on a viscous time scale. For this particular value of the viscosity parameter ($\alpha = 0.1$) the viscous instability does not induce oscillations in the inner part of the disc and the radial infall velocity stays subsonic (Papaloizou & Stanley 1986, see also Paczyński 1991). Moreover, since at the inner boundary the stellar surface rotates at a rate $\Omega_* = 0.98 \Omega_K$, the inner part of the disc is sustained by the centrifugal force, and the radial infall velocity in the boundary layer is not much different than in the outer disc. This situation is quite different from the one in which the angular velocity drops in the boundary layer and the inner part of the disc is sustained by the pressure force only. In this latter case the radial infall velocity in the boundary layer is very large. We assumed that at the outer stellar envelop $\Omega_* = 0.98 \Omega_K$

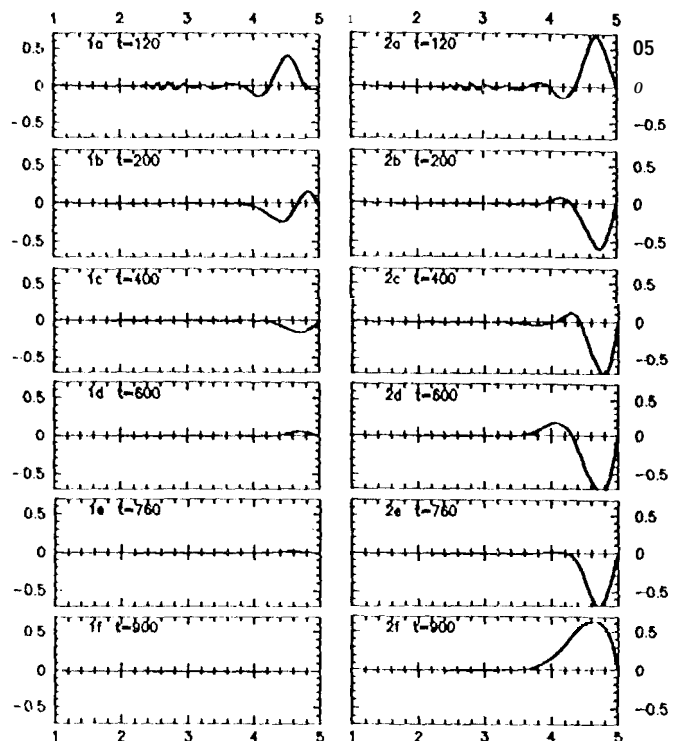


Figure 2. The Mach number is shown as a function of the radius for model 1 (left) and 2 (right). Six snapshots are taken at different times to show the evolution of models on a viscous time scale. In this case the outer boundary has been placed at $r = 5R_*$.

in order to avoid to deal with the numerical difficulties of a large gradient of the velocity in the inner part of the disc. This explains why in both models the Mach number in the inner part of the disc is not much different than in the rest of the disc. This has no effect on the results in the outer part of the disc and is somewhat equivalent to solving for a disc without BL, assuming for example $\partial\Omega/\partial r = 0$ at $r = R_*$.

In the inner part of the disc, oscillations due to the relaxation of the models propagate outward and decay with time. In figure 1 we show the evolution of model 1 for $t < 100$. For this particular case we have placed the outer boundary at $R_{\text{out}} = 2R_*$. The oscillations exit the outer boundary without being reflected. The numerical scheme is very stable and no instability appears at the boundary.

In order to study the effect of the boundary conditions on the viscous instability, we place the outer boundary at $R_{\text{out}} = 5R_*$. We expect the oscillations to appear in the outer part of the disc ($r > 3R_*$, Papaloizou & Stanley 1986, Okuda et al. 1992, Okuda & Mineshige 1991). The Mach number v_r/c_s is shown for models 1 and 2 in figures 2, at different times. The unit of time is the dynamic time. At the beginning of the evolutions, small amplitude oscillations propagate outward from the inner part of the disc, due to the relaxation of the models (like in figure 1). In the outer part of the disc, high amplitude oscillations appear. In model 1, the amplitude of the oscillations decays on a viscous time scale. However, in model 2 the amplitude of the oscillations does not change with time. At $t = 23900$, the amplitude of the

oscillations in model 1 is less than 0.01 while in model 2 it is still more than 0.5.

It was already pointed out by Papaloizou & Stanley (1986) that the viscous instability was found in the context of local analysis (Kato 1978 and Blumenthal et al. 1984). In the global analysis of Papaloizou & Stanley (1986) the viscous instability is expected to appear in the outer part of the disc, when the outer boundary is reflective. In this case the oscillations can be trapped between the outer boundary and the outer edge of the inner evanescent region. However, no non-reflective boundary conditions were imposed to actually verify the global effect of boundary conditions on the instability. In this work we have proven numerically that reflective boundary conditions lead to oscillations, while non-reflective boundary conditions lead to a stable disc, in which the oscillations decay on a viscous time scale. Furthermore, we have developed a treatment for non-reflective boundary conditions for a polytropic flow, while for more realistic flows, an approximation is carried out in the Appendix. This work is an additional proof that local instabilities (and in fact instabilities of any sort) in accretion discs, do not especially imply that a disc will be unstable globally. For the present case, the viscous instability in the disc will appear if a physical reflective outer boundary exists. This boundary can be, for example, a high density region, like a high density spiral arms (Godon 1995) formed by the tidal influence of a companion star (Savonije et al. 1994).

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APPENDIX A: NON-REFLECTIVE 1D UNIDIRECTIONAL CONDITIONS FOR A GAS WITH RADIATION

In this appendix, we treat non-reflective boundary conditions for a general flow including gas and radiation.

The equation of state is

$$P = \frac{R}{\mu} \rho T + \frac{aT^4}{3}. \quad (A1)$$

The homogeneous system of equations which represents the flow can be written:

$$\frac{\partial \vec{X}}{\partial t} + A \frac{\partial \vec{X}}{\partial r} = 0, \quad (A2)$$

where

$$\vec{X} = \begin{bmatrix} v_r \\ v_\phi \\ p \\ \rho \end{bmatrix}, \quad A = \begin{bmatrix} v_r & p & 0 & 0 \\ 0 & v_r & 0 & 1 \\ 0 & 0 & v_r & 0 \\ 0 & \gamma P & \sigma & v_r \end{bmatrix},$$

and γ is the ratio of the specific heat ($\gamma = \Gamma_1$). The matrix A can be diagonalized

$$S_H^{-1} A S_H = \begin{bmatrix} v_r & 0 & 0 & 0 \\ 0 & v_r + c_s & 0 & 0 \\ 0 & 0 & v_r & 0 \\ 0 & 0 & 0 & v_r - c_s \end{bmatrix}, \quad (A3)$$

with

$$S_H = \begin{bmatrix} \beta \rho & p & 0 & p \\ 0 & c_s & 0 & -c_s \\ 0 & 0 & \sqrt{2} c_s & 0 \\ 0 & \rho c_s^2 & 0 & \rho c_s^2 \end{bmatrix},$$

and

$$S_H^{-1} = \begin{bmatrix} 1/\beta \rho & 0 & 0 & -1/\beta \rho c_s^2 \\ 0 & 1/2c_s & 0 & 1/2\rho c_s^2 \\ 0 & 0 & 1/\sqrt{2}c_s & 0 \\ 0 & -1/2c_s & 0 & 1/2\rho c_s^2 \end{bmatrix}$$

The sound speed is $c_s^2 = \gamma P / \rho$ and $\beta = \sqrt{2}(\gamma - 1)$. However, since the following inequalities hold

$$\begin{cases} S_H^{-1} \frac{\partial \vec{X}}{\partial t} \neq \frac{\partial}{\partial t} (S_H^{-1} \vec{X}) \\ S_H^{-1} \frac{\partial \vec{X}}{\partial r} \neq \frac{\partial}{\partial r} (S_H^{-1} \vec{X}) \end{cases}, \quad (A4)$$

one has to linearize the equations. Therefore, we let $P = P_0 + \delta P$, $\rho = \rho_0 + \delta \rho$, $v_r = v_{r0} + \delta v_r$ and $v_\phi = v_{\phi 0} + \delta v_\phi$.

δv_ϕ , where $(\rho_0, v_{r0}, v_{\phi 0}, P_0)$ are the steady state solutions (for example the standard Shakura-Sunyaev thin disc) and $(\delta\rho, \delta v_r, \delta v_\phi, \delta P)$ are the perturbations, for which the system is solved. The characteristics equations become:

$$\left\{ \begin{array}{ll} \left[\frac{\partial}{\partial t} + v_r \frac{\partial}{\partial r} \right] \left(\frac{\delta\rho}{\rho_0 + \delta\rho} - \frac{1}{\gamma} \frac{\delta P}{P_0 + \delta P} \right) = 0, & \text{(i)} \\ \left[\frac{\partial}{\partial t} + v_r \frac{\partial}{\partial r} \right] \left(\frac{\delta v_\phi}{c_s} \right) = 0, & \text{(ii)} \\ \left[\frac{\partial}{\partial t} + (v_r + c_s) \frac{\partial}{\partial r} \right] \left(\frac{1}{\gamma} \frac{\delta P}{P_0 + \delta P} + \frac{\delta v_r}{c_s} \right) = 0, & \text{(iii)} \\ \left[\frac{\partial}{\partial t} + (v_r - c_s) \frac{\partial}{\partial r} \right] \left(\frac{1}{\gamma} \frac{\delta P}{P_0 + \delta P} - \frac{\delta v_r}{c_s} \right) = 0. & \text{(iv)} \end{array} \right. \quad (\text{A5})$$

The slow characteristics (i) & (ii) are incoming for $v_r < 0$ and outgoing for $v_r > 0$, (iii) is the fast outgoing characteristics and (iv) is the fast incoming characteristics. When $-c_s < v_r < 0$, one solves the system (i) = (i)_c, (ii) = (ii)_c, (iii) = (iii)_c and (iv) = (iv)_c; while when $c_s > v_r > 0$, one solves (i) = (i)_c, (ii) = (ii)_c, (iii) = (iii)_c and (iv) = (iv)_c. The above treatment was applied to two-dimensional problems and led to non-reflecting boundary conditions (Godon & Shaviv 1995).

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